CONCOURS D'ADMISSION 2017 FILIÈRE UNIVERSITAIRE INTERNATIONALE
AUTUMN SESSION

MATHEMATICS

(Duration : 2 hours)

The three parts (Exercises I and II, Problem) are independant.
The use of computing devices is not allowed

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Exercise 1

Compute the value of the integral

$$\int_x^{x^2} \frac{dt}{t \ln t}.$$

for the values of the real number x for which it is defined. One may use a change of variable.

Exercise 2

We denote by $\mathcal{C}_0([0,+\infty))$ the set of bounded continuous functions from $[0,+\infty)$ to $\mathbb{R}$. We let a be a non-negative real number and $h \in \mathcal{C}_0([0,+\infty))$. In this exercise, we study the differential equation

$$f' + af = h \text{ denoted by } (E_a(h)),$$

where the unknown f is a differentiable function from $(0,+\infty)$ to $\mathbb{R}$.

Question 1. For any $a \geq 0$ determine all the solutions of the equation $f' + af = 0$.

Question 2. For $a \geq 0$ and $x \in (0,+\infty)$ we let

$$\varphi_a(x) = \int_0^x h(t) e^{+a(t-x)} dt.$$

Compute the derivative of φ_a.

Question 3. Using the function φ_a, determine all the solutions of $(E_a(h))$.

Question 4. Determine all the values of $a \geq 0$ such that for any $h \in \mathcal{C}_0([0, +\infty))$ all the solutions of the equation $(E_a(h))$ are bounded.

Question 5. Find a necessary and sufficient condition for h such that for any $a \geq 0$ all the solutions of the equation $(E_a(h))$ are bounded.

PROBLEM

GENERAL NOTATION. A vector $x \in \mathbb{R}^3$ is a column 3×1 matrix, which will also be represented as the transposed of an array 1×3 matrix, as

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = {}^t(x_1, x_2, x_3).$$

We denote by $\{e^{(1)}, e^{(2)}, e^{(3)}\}$ the natural basis of $\mathbb{R}^3$, with $e_i^{(j)} = 1$ when $i = j$ and $e_i^{(j)} = 0$ when $i \neq j$.

In this problem, we study properties of the 3×3 real matrix

$$M = \begin{pmatrix} 0.5 & 0.3 & 0.2 \\ 0.2 & 0.4 & 0.5 \\ 0.3 & 0.3 & 0.3 \end{pmatrix}.$$

Question 1. Find the eigenvalues of the matrix

$$A = \begin{pmatrix} 5 & 3 & 2 \\ 2 & 4 & 5 \\ 3 & 3 & 3 \end{pmatrix}.$$

Question 2. Using Question 1., find the eigenvalues of the matrix M. They will be denoted by λ_1, λ_2 and λ_3, with $|\lambda_1| > \max(|\lambda_2|, |\lambda_3|)$, the choice between λ_2 and λ_3 being left to the candidate.

Question 3 Compute a non zero vector $\varepsilon^{(1)}$ in $\mathbb{R}^3$, eigenvector of M for the eigenvalue λ_1.

Question 4. Let $\varepsilon^{(2)}$ (*respectively* $\varepsilon^{(3)}$) be a non zero eigenvector for the eigenvalue λ_2 (*respectively* λ_3). Is the family $\{\varepsilon^{(1)}, \varepsilon^{(2)}, \varepsilon^{(3)}\}$ a basis of $\mathbb{R}^3$?

From now on we consider a vector

$$x^{(0)} = {}^t(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}) = y_1^{(0)}\varepsilon^{(1)} + y_2^{(0)}\varepsilon^{(2)} + y_3^{(0)}\varepsilon^{(3)},$$

we define by induction

$$x^{(n)} = Mx^{(n-1)}, \text{ for } n \geq 1$$

and we write

$$x^{(n)} = {}^t(x_1^{(n)}, x_2^{(n)}, x_3^{(n)}) = y_1^{(n)}\varepsilon^{(1)} + y_2^{(n)}\varepsilon^{(2)} + y_3^{(n)}\varepsilon^{(3)}.$$

Question 5. Show that for any $j \in \{1, 2, 3\}$, the sequence $\left(y_j^{(n)}\right)_{n \geq 0}$ is convergent and determine its limit.

Question 6. Deduce from the previous question that for any $j \in \{1, 2, 3\}$, the sequence $\left(x_j^{(n)}\right)_{n \geq 0}$ is convergent and determine its limit.

Question 7. Show that for any integer n the sum of the entries of each column of the matrix M^n is equal to 1.

Question 8. Show that the sequence of the powers of the matrix M (that is to say the sequence $(M^n)_n$) converges to a matrix all the columns of which are equal. We recall that a sequence of matrices $B^{(n)} = (b_{i,j}^{(n)})_{i,j}$ converges if for any pair (i, j), the sequence $(b_{i,j}^{(n)})_n$ converges.

Question 9. Determine explicitly the limit of the sequence $(M^n)_n$.

Session 2

CONCOURS D'ADMISSION 2017 FILIÈRE UNIVERSITAIRE INTERNATIONALE

MATHEMATICS

(Duration : 2 hours)

The three parts (I, II and III) are independant.
The use of computing devices is not allowed

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Part I

We consider the matrix

$$A = \begin{pmatrix} -1 & 2 \\ -3 & 4 \end{pmatrix}.$$

Question I.1. Determine the eigenvalues of the matrix A.

Question I.2. Determine two eigenvectors respectively associated to the two eigenvalues of A.

From now on, we consider two real numbers u_0 and u_1 and we define by induction the sequence $(u_n)_{n\geq 0}$ thanks to the following relations, valid for any $n \geq 1$:

$$u_{2n} = 2u_{2n-1} - u_{2n-2} \qquad \text{and} \qquad u_{2n+1} = 3u_{2n} - 2u_{2n-1}.$$

Question I.3. For $n \geq 0$, we let $X_n = \begin{pmatrix} u_{2n} \\ u_{2n+1} \end{pmatrix}$. Express X_{n+1} in terms of X_n and A.

Question I.4. In this question, we assume that $u_0 = u_1 = 1$. Compute the vectors X_n associated to this choice of u_0 and u_1.

Question I.5. Give a necessary and sufficient condition on the pair (u_0, u_1) for the sequence $(X_n)_{n\geq 0}$ to be bounded.

Part II

Let f be a continuous function from $[0, +\infty)$ into $\mathbb{R}$, strictly decreasing and integrable on $[0, +\infty)$. We recall that saying that the function f is integrable on $[0, +\infty)$ means that the quantity $\int_0^x |f(t)|dt$ has a limit when x tends to ∞.

Question II.1. Show that the limit

$$\lim_{x \to +\infty} f(x)$$

exists and is equal to zero.

Question II.2. Show that for all $x \geq 0$ one has $f(x) > 0$.

Question II.3. Show that for all integers $k \geq 1$ one has

$$\int_k^{k+1} f(t)dt \ < \ f(k) \ < \ \int_{k-1}^{k} f(t)dt.$$

Question II.4. Deduce from the previous questions that the series, the general term of which is $f(k)$, converges and that one has

$$\forall n \geq 1 : \ \int_n^{+\infty} f(t)dt \ < \sum_{k=n}^{+\infty} f(k) \ < \ \int_{n-1}^{+\infty} f(t)dt.$$

Question II.5. For $\alpha \in \mathbb{R}$ and $t \in [0, +\infty)$ let $f_\alpha(t) = 1/(t+1)^\alpha$.

II.5.a Determine the values of α for which the functionf_α is integrable over $[0, +\infty)$.

II.5.b We assume that the function f_α is integrable over $[0, +\infty)$. Does the following limit exist, and if it exists, what is its value ?

$$\lim_{n \to +\infty} \left(\sum_{k=n}^{+\infty} f_\alpha(k) \right) \left(\int_n^{+\infty} f_\alpha(t)dt \right)^{-1}.$$

Question II.6. Does the following limit exist, and if it exists, what is its value ?

$$\lim_{n \to +\infty} \left(\sum_{k=n}^{+\infty} \exp(-k) \right) \left(\int_n^{+\infty} \exp(-t)dt \right)^{-1}.$$

Part III

We consider two polynomials P and Q with complex coefficients.

Question III.1. We assume that the polynomials $P + Q$ and $P - Q$ are constant. Show that the polynomials P and Q are constant.

Question III.2. We assume that the polynomial $P^2 - Q^2$ is a non zero constant. Show that the polynomials P and Q are constant.

Question III.3. Let $n \geq 2$ be an integer. We assume that the polynomial $P^n - Q^n$ is a non zero constant. Show that the polynomials P and Q are constant. Hint : one can use the factorisation of the polynomial $X^n - 1$ in $\mathbb{C}[X]$, in terms of the n-th roots of unity.

Session 2

CONCOURS D'ADMISSION 2017 FILIÈRE UNIVERSITAIRE INTERNATIONALE

MATHÉMATIQUES

(Durée : 2 heures)

Les trois parties (I, II et III) sont indépendantes
L'utilisation de moyens de calcul est interdite

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Partie I

On considère la matrice

$$A = \begin{pmatrix} -1 & 2 \\ -3 & 4 \end{pmatrix}.$$

Question I.1. Déterminer les valeurs propres de la matrice A.

Question I.2. Déterminer deux vecteurs propres associés respectivement à chaque valeur propre de A.

On considère dans la suite deux nombres réels u_0 et u_1 et on définit par récurrence la suite $(u_n)_{n\geq 0}$ à l'aide des relations suivantes, valables pour tout $n \geq 1$:

$$u_{2n} = 2u_{2n-1} - u_{2n-2} \qquad \text{et} \qquad u_{2n+1} = 3u_{2n} - 2u_{2n-1}.$$

Question I.3. Pour $n \geq 0$, on pose $X_n = \begin{pmatrix} u_{2n} \\ u_{2n+1} \end{pmatrix}$. Exprimer X_{n+1} en fonction de X_n et A.

Question I.4. Dans cette question, on suppose que $u_0 = u_1 = 1$. Calculer les vecteurs X_n associés à ce choix de u_0 et u_1.

Question I.5. Donner une condition nécessaire et suffisante sur le couple (u_0, u_1) pour que la suite $(X_n)_{n\geq 0}$ soit bornée.

Partie II

Soit f une fonction continue de $[0, +\infty[$ dans $\mathbb{R}$, strictement décroissante et intégrable sur $[0, +\infty[$. On rappelle que dire que la fonction f est intégrable sur $[0, +\infty[$ équivaut à dire que l'expression $\int_0^x |f(t)|dt$ a une limite quand x tend vers l'infini.

Question II.1. Montrer que la limite

$$\lim_{x\to+\infty} f(x)$$

existe et est égale à zéro.

Question II.2. Montrer que pour tout $x \geq 0$ on a : $f(x) > 0$.

Question II.3. Montrer que pour tout entier $k \geq 1$ on a

$$\int_k^{k+1} f(t)dt \;<\; f(k) \;<\; \int_{k-1}^{k} f(t)dt.$$

Question II.4. Déduire de ce qui précède que la série de terme général $f(k)$ converge et que l'on a

$$\forall n \geq 1 : \int_n^{+\infty} f(t)dt \;<\; \sum_{k=n}^{+\infty} f(k) \;<\; \int_{n-1}^{+\infty} f(t)dt.$$

Question II.5. Pour $\alpha \in \mathbb{R}$ et $t \in [0, +\infty[$ on pose $f_\alpha(t) = 1/(t+1)^\alpha$.

II.5.a Déterminer les valeurs de α telles que la fonction f_α soit intégrable sur $[0, +\infty[$.

II.5.b On suppose que la fonction f_α est intégrable sur $[0, +\infty[$. La limite suivante existe-t-elle, et si elle existe, quelle est sa valeur ?

$$\lim_{n\to+\infty} \left(\sum_{k=n}^{+\infty} f_\alpha(k)\right) \left(\int_n^{+\infty} f_\alpha(t)dt\right)^{-1}.$$

Question II.6. La limite suivante existe-t-elle, et si elle existe, quelle est sa valeur ?

$$\lim_{n\to+\infty} \left(\sum_{k=n}^{+\infty} \exp(-k)\right) \left(\int_n^{+\infty} \exp(-t)dt\right)^{-1}.$$

Partie III

On considère deux polynômes P et Q à coefficients complexes.

Question III.1. On suppose que les polynômes $P + Q$ et $P - Q$ sont constants. Montrer que les polynômes P et Q sont constants.

Question III.2. On suppose que le polynôme $P^2 - Q^2$ est constant et non nul. Montrer que les polynômes P et Q sont constants.

Question III.3. Soit n un entier supérieur à 2. On suppose que le polynôme $P^n - Q^n$ est constant non nul. Montrer que les polynômes P et Q sont constants. Indication : on pourra utiliser la factorisation du polynôme $X^n - 1$ dans $\mathbb{C}[X]$, à l'aide des racines n-ièmes de l'unité.