ÉCOLE POLYTECHNIQUE
UNIVERSITÉ PARIS-SACLAY

CONCOURS D'ADMISSION 2016 – FILIÈRE UNIVERSITAIRE INTERNATIONALE
SESSION AUTOMNE 2015

MATHEMATICS

(Duration : 2 hours)

⋆ ⋆ ⋆

This exam is composed of 5 exercises. One can solve them in *any order.* Generally speaking, the first questions are simpler and the final questions are more difficult. One should therefore *not spend too much time* on these final questions before having solved the other exercices.

In solving a given exercise, one is allowed to use the results of the preceding questions *(including those one could not prove).* All statements must be *clearly* and *completely* justified.

I.

Let P be a polynomial in $\mathbb{R}[X]$ satisfying $(*)$: $P(P(X)) = P(X)^2$.

1. Are there constant polynomials solutions to $(*)$?

From now on, we assume that P is a non-constant polynomial solution to $(*)$.

2. Determine the degree of P.

3 What is the value of the coefficient of X^2 in P ?

4. Find all the polynomials P solutions to $(*)$?

5. By an analogous reasoning, find the solutions $P \in \mathbb{C}[X]$ of the equation :

$$P(P''(X)) = (P(X+1) - P(X) - P'(X))^3 .$$

II.

Let n be a positive integer, E be an n-dimensional $\mathbb{C}$-vector space and $(e_1, \dots, e_n)$ be a basis of E. Let u be the endomorphism of E defined by the relations :

$$\text{for all } i \in \{1, \dots, n\} : u(e_i) = e_i + f$$

where $f = \sum_{k=1}^{n} e_k$.

1. What is the matrix of u in the basis $(e_1, \ldots, e_n)$? We call it U in the following questions.

2. Is the matrix U invertible ?

3. Let J be the matrix which has ones in all positions ($J_{i,j} = 1$ for all $1 \leq i, j \leq n$). Compute J^2.

4. Find the eigenvalues of U. For each of them, give a basis of the associated eigenspace.

5. Compute, for each non-negative integer m, the m-th power of U as a function of J.

III.

Let M_0 be the 3×3 square matrix :

$$M_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

1. Compute ${M_0}^3$ in terms of M_0.

Let now M be a 3×3 square matrix with real coefficients, $M \in \mathcal{M}(\mathbb{R}^3)$, such that

$$M^3 = -M .$$

We assume $M \neq 0$.

2. Is the matrix M diagonalizable as a real matrix? What about diagonalizability as a complex matrix?

From now on, we are only interested in the reduction of M as a real matrix.

3. Show that

$$\mathbb{R}^3 = \text{Ker } M \oplus \text{Ker } (M^2 + I) ,$$

where I stands for the identity matrix in dimension 3.

4. Prove that the dimension of Ker $(M^2 + I)$ is even. Then deduce dim Ker $M = 1$.

5. Show that there exists a vector x in $\mathbb{R}^3$ such that the family $\{x, Mx\}$ is free.

6. Prove that M is similar to M_0.

IV.

In this exercise, we are interested in the solutions to the equation (E) : $\tan x = x$.

1. Show that for each integer n, (E) has exactly one solution in the interval $(n\pi - \pi/2, n\pi + \pi/2)$. We denote it by x_n.

2. Prove that, when n tends to infinity, $x_n - n\pi$ tends towards $\pi/2$.

3. Compute the limit of the quantity $v_n = n(x_n - n\pi - \pi/2)$ when n tends to infinity.

V.

We define the functions

$$I(x) = \int_0^{+\infty} \frac{e^{-t}}{\sqrt{t}} \cos tx \, dt \qquad \text{and} \qquad J(x) = \int_0^{+\infty} \frac{e^{-t}}{\sqrt{t}} \sin tx \, dt \, .$$

1. Explain why I and J are well defined on $\mathbb{R}$.

2. Show that I and J are differentiable and compute their derivatives I' and J'.

3. Integrating by parts, prove that

$$I'(x) = -\frac{1}{2}J(x) - xJ'(x) \, .$$

In the same way, find a relation between J', I and I'.

4. Deduce that I and J both satisfy a differential equation of the type

$$2(1+x^2)y' + xy = f$$

where f is a certain function (not necessarily the same for I and J).

5. Solve the preceding system and determine I and J in terms of x uniquely.

* *
*

CONCOURS D'ADMISSION 2016 – FILIÈRE UNIVERSITAIRE INTERNATIONALE
SESSION DE PRINTEMPS

MATHEMATICS

(Duration : 2 hours)

* * *

This exam is composed of four exercises. One can solve them *in any order.* It is not needed to solve all the exercises to obtain the best possible mark.

Generally speaking, the difficulty of the questions is an increasing function of their number so that *the final questions are more difficult.* It is therefore not recommended to spend too much time on the final questions of an exercise before having solved the first questions of the others.

In solving a given exercise, one is allowed to use the results of the preceding questions *(including those one could not prove).*

All statements must be *clearly* and *completely* justified.

I. Matrices with a robust diagonal

A matrix $M \in \mathcal{M}_n(\mathbb{C})$ is said to have a robust diagonal if the elements on its diagonal coincide with its eigenvalues (with the same multiplicities). We denote by $\mathcal{D}_n$ the subset of $\mathcal{M}_n(\mathbb{C})$ composed of all the matrices with a robust diagonal.

1) Identify $\mathcal{D}_1$.

2) Show that every matrix in $\mathcal{M}_n(\mathbb{C})$ is similar to a matrix with a robust diagonal.

3) Identify $\mathcal{D}_2$. Is this set open ? closed ? path-connected ? convex ?

4) For a given integer n, is the set of all matrices with a robust diagonal $\mathcal{D}_n$ a vector space ?

5) Determine, among the real symmetric matrices, those belonging to $\mathcal{D}_n$. (Hint : one may introduce the quantity $\mathrm{tr}({}^tMM)$).

II. Study of a sequence

Let c be a non-negative real number and $(u_n)_{n\geq 1}$ be the sequence defined by $u_1 = 1$ and the induction relation

$$u_{n+1} = \sqrt{u_n + cn}.$$

1) Find a real number σ such that for any integer $n \geq 1$, one has $u_n \leq \sigma\sqrt{n}$.

2) Find an equivalent of u_n, when n tends to infinity, of the form $u_n \sim \alpha n^\beta$ (where α and β are real numbers).

3) Compute the limit of $u_n - \alpha n^\beta$ as n tends to infinity.

III. Homogeneous polynomials

For a given positive integer n, we denote by $\mathcal{H}_n \subset \mathbb{R}[X,Y]$ the set of homogeneous polynomials with real coefficients of degree n in two variables (X and Y). We recall that these are the polynomials P having the property that $P(\lambda X, \lambda Y) = \lambda^n P(X,Y)$ for any real number λ.

We denote by $\mathcal{D}_n$ the subset of $\mathcal{H}_n$ composed of the polynomials divisible by $X^2 + Y^2$ and we denote by $\mathcal{L}_n$ the subset of $\mathcal{H}_n$ composed of those polynomials P satisfying

$$\frac{\partial^2 P}{\partial X^2} + \frac{\partial^2 P}{\partial Y^2} = 0.$$

1) Prove that, for any integer n, $\mathcal{H}_n$, $\mathcal{D}_n$ and $\mathcal{L}_n$ are vector spaces.

2) Prove that any polynomial in $\mathcal{H}_n$ can be written as

$$\sum_{k=0}^{n} c_k X^k Y^{n-k}$$

where the c_k's are real numbers. What is the dimension of $\mathcal{H}_n$?

3) What are the relations that the coefficients of P must satisfy if $P \in \mathcal{L}_n$?

4) Using question 3), give a basis of $\mathcal{L}_n$.

5) Prove that $\mathcal{D}_n \cap \mathcal{L}_n = \{0\}$.

6) Prove that $\dim \mathcal{D}_n = \dim \mathcal{H}_{n-2}$.

7) Prove that $\mathcal{D}_n \oplus \mathcal{L}_n = \mathcal{H}_n$.

IV. Study of a power series

1) For which integers n does the equality $\sin(n\pi\sqrt{5}) = 0$ hold ?

Let R be the radius of convergence of the power series $\displaystyle\sum_{n\geq 1} \frac{z^n}{\sin(n\pi\sqrt{5})}$.

2) Prove that $R \leq 1$.

3) Prove that, for all $t \in [0, \pi/2]$, one has $\sin t \geq t - t^3/6$.

4) Prove that for any integers p and q, non simultaneously equal to zero, one has

$$|p\sqrt{5} - q| \geq \frac{1}{p\sqrt{5} + q}.$$

5) Using questions 3) and 4), find a real number $c > 0$ such that, for all integers $n \geq 1$, one has

$$|\sin(n\pi\sqrt{5})| \geq \frac{c}{n}.$$

6) Compute R.

CONCOURS D'ADMISSION 2016 – FILIÈRE UNIVERSITAIRE INTERNATIONALE
SESSION DE PRINTEMPS

MATHÉMATIQUES

(Durée : 2 heures)

* * *

Ce sujet est composé de quatre exercices. On peut les traiter dans *n'importe quel ordre.* Il n'est pas nécessaire de résoudre tous les exercices pour obtenir la note maximale.

D'une façon générale, la difficulté des questions est une fonction croissante de leur numéro de sorte que *les questions finales sont les plus dures.* On ne doit donc pas consacrer trop de temps aux dernières questions d'un exercice avant d'avoir résolu les premières questions des autres.

Dans la résolution d'un exercice donné, on peut toujours utiliser le résultat des questions précédentes *(même si on ne les a pas résolues).*

Toute affirmation doit être *clairement* et *complètement* justifiée.

I. Matrices à diagonale solide

On dit qu'une matrice $M \in \mathcal{M}_n(\mathbb{C})$ est à diagonale solide si les éléments de sa diagonale sont ses valeurs propres (avec mêmes multiplicités). On note $\mathcal{D}_n$ le sous-ensemble de $\mathcal{M}_n(\mathbb{C})$ composé de toutes les matrices à diagonale solide.

1) Identifier $\mathcal{D}_1$.

2) Montrer que toute matrice de $\mathcal{M}_n(\mathbb{C})$ est semblable à une matrice à diagonale solide.

3) Identifier $\mathcal{D}_2$. Cet ensemble est-il ouvert ? fermé ? connexe par arcs ? convexe ?

4) Pour n un entier donné, l'ensemble de toutes les matrices à diagonale solide $\mathcal{D}_n$ est-il un espace vectoriel ?

5) Déterminer les matrices symétriques réelles M appartenant à $\mathcal{D}_n$. (Indication : on pourra introduire la quantité $\mathrm{tr}({}^tMM)$).

II. Étude d'une suite

Soit c un nombre réel positif et $(u_n)_{n\geq 1}$ la suite définie par $u_1 = 1$ et la relation de récurrence
$$u_{n+1} = \sqrt{u_n + cn}.$$

1) Trouver un réel σ tel que pour tout entier $n \geq 1$, on ait $u_n \leq \sigma\sqrt{n}$.

2) Trouver un équivalent de u_n, lorsque n tend vers l'infini, de la forme $u_n \sim \alpha n^\beta$ (où α et β sont des nombres réels).

3) Calculer la limite de $u_n - \alpha n^\beta$ lorsque n tend vers l'infini.

III. Polynômes homogènes

Pour n un entier strictement positif quelconque, on note $\mathcal{H}_n \subset \mathbb{R}[X,Y]$ l'ensemble des polynômes homogènes à coefficients réels de degré n en deux variables (X et Y) : on rappelle qu'il s'agit des polynômes P ayant la propriété que $P(\lambda X, \lambda Y) = \lambda^n P(X,Y)$ quel que soit le réel λ.

On note $\mathcal{D}_n$ le sous-ensemble de $\mathcal{H}_n$ composé des polynômes divisibles par $X^2 + Y^2$ et par $\mathcal{L}_n$ celui des polynômes P de $\mathcal{H}_n$ tels que
$$\frac{\partial^2 P}{\partial X^2} + \frac{\partial^2 P}{\partial Y^2} = 0.$$

1) Démontrer que, quel que soit l'entier n, $\mathcal{H}_n$, $\mathcal{D}_n$ et $\mathcal{L}_n$ sont des espaces vectoriels.

2) Démontrer que tout polynôme de $\mathcal{H}_n$ peut s'écrire sous la forme
$$\sum_{k=0}^{n} c_k X^k Y^{n-k}$$
où les c_k sont des réels ? Quelle est la dimension de $\mathcal{H}_n$?

3) Quelles relations doivent vérifier les coefficients de P si $P \in \mathcal{L}_n$?

4) Déduire de la question 3) une base de $\mathcal{L}_n$.

5) Démontrer que $\mathcal{D}_n \cap \mathcal{L}_n = \{0\}$.

6) Démontrer que $\dim \mathcal{D}_n = \dim \mathcal{H}_{n-2}$.

7) Démontrer que $\mathcal{D}_n \oplus \mathcal{L}_n = \mathcal{H}_n$.

IV. Étude d'une série entière

1) Pour quels entiers n a-t-on $\sin(n\pi\sqrt{5}) = 0$?

Soit R le rayon de convergence de la série entière $\displaystyle\sum_{n\geq 1} \frac{z^n}{\sin(n\pi\sqrt{5})}$.

2) Démontrer que $R \leq 1$.

3) Démontrer que, pour tout $t \in [0, \pi/2]$, on a : $\sin t \geq t - t^3/6$.

4) Démontrer que quels que soient les entiers p et q, non nuls simultanément, on a

$$|p\sqrt{5} - q| \geq \frac{1}{p\sqrt{5} + q}.$$

5) En utilisant les questions 3) et 4), trouver un réel $c > 0$ tel que, pour tout entier $n \geq 1$, on ait

$$|\sin(n\pi\sqrt{5})| \geq \frac{c}{n}.$$

6) Que vaut R?