

CONCOURS D'ADMISSION 2017
AUTUMN SESSION

FILIÈRE UNIVERSITAIRE INTERNATIONALE

PHYSICS

(Duration : 2 hours)

*The problem and the exercise are independent and can be solved in any order.
Make sure to carry out numerical calculations when requested and round numerical values to one significant digit. The use of electronic calculators is forbidden.*

Problem – Water purification

The purpose of this problem is to study the removal of small solid particles from liquid water as part of the water purification process. We study two methods used for this purpose : decantation (questions 1 to 4) and coagulation (questions 5 to 16).

Decantation consists in waiting that the particles fall to the bottom of the container through the action of gravity which acceleration is denoted by \vec{g} .

1. Use Archimedes' principle to evaluate the buoyant (upward) force \vec{F}_A exerted on a solid particle of volume V and density (volumetric mass density) μ_s surrounded by a static liquid of density μ . Under what condition does the particle fall to the bottom ?

A spherical particle of radius R falling with velocity \vec{v} in a fluid of kinematic viscosity ν experiences a drag force given by Stokes' law : $\vec{F}_d = -6\pi\nu\mu R\vec{v}$.

2. What is the dimension of ν ?

3. Show that the falling particle reaches a maximum velocity v_d , the so-called decantation velocity. Express v_d as a function of R , ν , g and of the ratio μ_s/μ .

4. What condition must v_d satisfy in order for decantation to be efficient after a duration $\Delta t = 1$ day in a container of height $h = 1$ m ? What is the resulting condition on the radius R of particles whose density is twice that of water, $\mu_s = 2\mu$? Express the result first algebraically, then numerically ; kinematic viscosity of water is $\nu = 10^{-6}$ SI.

We now study an alternative process named coagulation, which is the aggregation of small particles into larger particles, which are eventually eliminated by decantation. Aggregation is triggered by the van der Waals attraction between two neighboring particles.

But this attraction is counterbalanced by the electrostatic repulsion between particles, which carry an electric charge on their surface. We assume for simplicity that the charge density per unit surface σ is positive and identical for all particles. In order to induce coagulation, one must first decrease this electrostatic repulsion. This is achieved by adding ions of opposite charges q et $-q$ to the water, with $q > 0$. The densities of positive and negative ions at infinity are identical and denoted by n (number density per unit volume).

5. Describe qualitatively how the concentrations of positive and negative ions are modified in the vicinity of the surface of a particle.

We now carry out a more quantitative study of the electric potential near the surface of a particle. We model this surface as an infinite plane Oyz . The particle occupies the half-space $x < 0$, the water occupies the half-space $x > 0$.

6. Write the differential equation relating the electric potential $V(x)$ to the charge density $\rho(x)$ inside the liquid.

Hint : The electric field \vec{E} is related to the charge density ρ by $\text{div} \vec{E} = \rho/\epsilon$, where ϵ denotes the dielectric permittivity of the water.

7. What is the potential energy $U_+(x)$ of a positive ion at point x ? What is the potential energy $U_-(x)$ of a negative ion at point x ?

8. Explain why the number density of positive or negative ions at point x is given by $n_{\pm}(x) = n \exp(-U_{\pm}(x)/k_B T)$, where T is the temperature of the solution.

9. Use this result to express the *total* charge density $\rho(x)$ as a function of $V(x)$.

10. Write the differential equation satisfied by $V(x)$ for $x > 0$. Linearize this equation when $|U_{\pm}(x)| \ll k_B T$. Show that a characteristic length λ appears in this equation.

11. Solve the differential equation for $V(x)$ and sketch its graph.

12. Use Gauss's flux theorem to express the electric field at $x = 0^+$ as a function of charge density per unit surface σ . Use this result to relate $V(x = 0^+)$ and σ . What is the electrical analog of this situation?

13. Explain how the differential equation satisfied by $V(x)$ can be solved if the condition $|U_{\pm}(x)| \ll k_B T$ is released. Describe your method without doing all the calculation.

Hint : Use a mechanical analogy with the equation of motion of a particle under a conservative force.

14. Using previous results, justify briefly why the electrostatic energy between two solid particles whose mutual distance x is much smaller than their size can be written as

$$U_E(x) = B \exp\left(-\frac{x}{\lambda}\right)$$

where B is a positive constant.

In addition, the particles interact through the Van der Waals force which associated potential energy is given by $U_V(x) = -A/x^2$ where A is a positive constant. The total interaction energy is thus $U(x) = U_E(x) + U_V(x)$.

15. Determine under which condition $U(x) \leq 0$ for all $x > 0$.

Hint : Introduce a dimensionless parameter β proportional to B and discuss the relative importance of each energy as a function of β .

16. Under which condition on β does coagulation occur?

Exercise – Nucleation

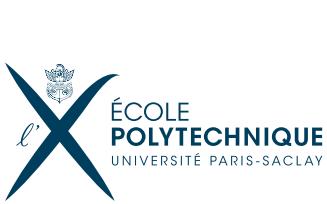
When the temperature of a liquid is lowered below its freezing point, it turns into a solid through the nucleation of small grains. While the transformation of the liquid into a solid is energetically favored below the freezing point, the formation of a grain comes with an additional cost due to the surface tension between the solid and the liquid. The purpose of this exercise is to study the competition between these two effects.

1. Recall why the Gibbs energy (or free enthalpy) G is the relevant thermodynamic potential at fixed pressure and temperature.
2. We denote by μ_S and μ_L the Gibbs energy per mole in the solid and liquid phase, respectively. What is the sign of $\mu_S - \mu_L$ below the freezing point ?

The density (volumetric mass density) of the solid phase is ρ_S and the molar mass is M . We recall that the surface tension γ of the interface between the solid and the liquid is defined as the work $dW = \gamma dA$ needed to increase its surface area by dA at fixed temperature and pressure.

3. Write the variation $\Delta G(R)$ associated with the nucleation of a spherical solid grain of radius R . Sketch its variation as a function of R .
4. Discuss the fate of a grain depending on its size.

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ÉCOLE POLYTECHNIQUE

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SESSION DU PRINTEMPS 2017

PHYSICS

(Duration : 2 hours)

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Short questions

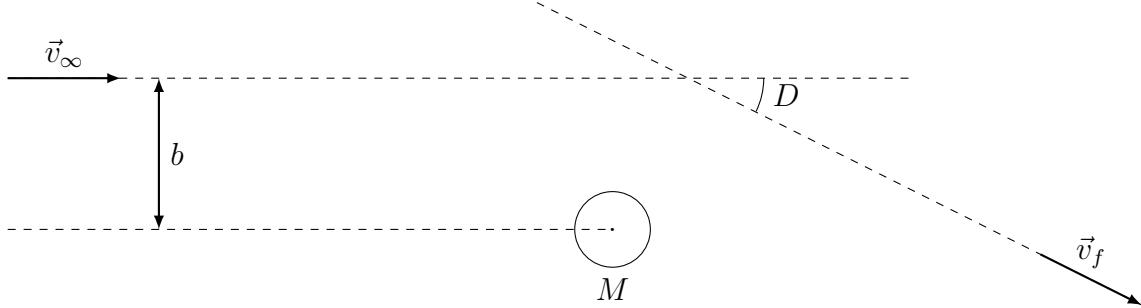
Answer each question the way you prefer, but always detail and explain your assumptions and calculations. Here are some useful physical values :

Gravitational constant	$G = 6.7 \times 10^{-11}$ SI
Boltzmann constant	$k = 1.4 \times 10^{-23}$ J K $^{-1}$
Avogadro constant	$N_A = 6.0 \times 10^{23}$ mol $^{-1}$
Earth mass	$M_{\oplus} = 6.0 \times 10^{24}$ kg
Earth radius	$R_{\oplus} = 6\,400$ km

1. Using dimensional analysis, calculate an order of magnitude of the pressure at the center of the Earth.
2. Considering that you can make a vertical jump of 50 cm on Earth, estimate the maximum size of an asteroid from which you can escape by jumping.
3. Estimate the mass of the Earth atmosphere.
4. Estimate the thickness of Earth atmosphere considered as isothermal with molar mass of 29 g/mol.
5. Determine the scaling of the speed of walking as a function of human size L assuming that a leg swings as a pendulum.

Exercise – Gravitational deflection

We study the gravitational deflection of a particle and build an optical analogy. The particle of mass m is sent from infinity with an impact parameter b and initial non relativistic velocity \vec{v}_∞ . It is deflected by a body of mass $M \gg m$ and radius R . We denote by (\vec{r}, \vec{v}) the general position and velocity of the particle with respect to the center of the body. After deflection, the final velocity of the particle at infinity is \vec{v}_f .



1. Show that the angular momentum \vec{L} of the particle is constant.
2. Show that the particle will strike the planet, at least at grazing incidence, if its impact parameter is such that $b \leq b_0$ where

$$b_0 = R \left(1 + \frac{r_\infty}{R} \right)^{1/2}$$

Express r_∞ as a function of M and v_∞ .

Hint : Write the conservation of angular momentum and total energy applied to the grazing trajectory.

3. Show that the Runge-Lenz vector $\vec{A} = \vec{v} \times \vec{L} - GMm \vec{r}/r$ is constant.

Hint : We recall that $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c}) \vec{b} - (\vec{a} \cdot \vec{b}) \vec{c}$.

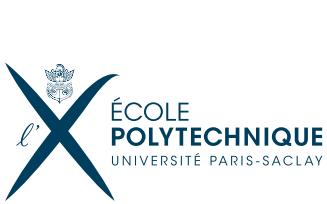
4. Use the previous result to show that when $b > b_0$ the deviation angle D between initial and final velocities of the particle is given by

$$\tan \left(\frac{D}{2} \right) = \frac{r_\infty}{2b}$$

Now we consider a lens of refractive index n , with a flat entry face perpendicular to its optical symmetry axis Oz . The lens is illuminated parallel to this axis. The exit face has a varying thickness $T(b)$ where b is the distance to the optical axis and $T(b_0) = T_0$.

5. Find how $T(b)$ must vary in order for the optical deviation to mimic the gravitational deflection in the limit of small deviation.

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PHYSIQUE

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Les exercices sont indépendants et peuvent être résolus dans l'ordre que vous souhaitez. N'oubliez pas de faire les application numériques demandées et arrondissez vos résultats à un chiffre après la virgule. L'usage d'une calculatrice électronique est interdit.

Questions courtes

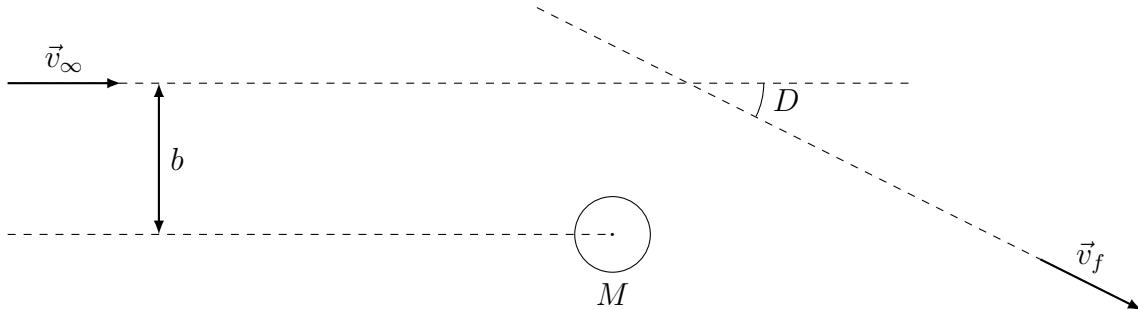
Répondez aux questions suivantes en explicitant vos hypothèses et en détaillant vos calculs. Voici quelques valeurs numériques utiles :

Constante de gravitation	$G = 6.7 \times 10^{-11}$ SI
Constante de Boltzmann	$k = 1.4 \times 10^{-23}$ J K $^{-1}$
Nombre d'Avogadro	$N_A = 6.0 \times 10^{23}$ mol $^{-1}$
Masse de la Terre	$M_{\oplus} = 6.0 \times 10^{24}$ kg
Rayon de la Terre	$R_{\oplus} = 6\,400$ km

1. En utilisant l'analyse dimensionnelle, calculez un ordre de grandeur de la pression qui règne au centre de la Terre.
2. Considérant que vous êtes capable de faire un saut vertical de 50 cm sur Terre, estimatez la taille maximum d'un astéroïde dont vous pourriez vous échapper en sautant.
3. Estimatez la masse de l'atmosphère terrestre.
4. Estimatez l'épaisseur de l'atmosphère terrestre considérée comme isotherme et constituée d'un gaz dont la masse molaire vaut 29 g/mol.
5. Déterminez la loi d'échelle liant la vitesse de marche d'un humain en fonction de sa taille L . Vous assimilerez le mouvement d'une jambe à celui d'un pendule simple.

Exercice – Déviation gravitationnelle

Nous allons étudier la déviation gravitationnelle et en construire une analogie optique. Une particule de masse m est envoyée de l'infini avec un paramètre d'impact b et une vitesse initiale non relativiste \vec{v}_∞ . Elle est déviée par un corps de masse $M \gg m$ et de rayon R . Vous noterez (\vec{r}, \vec{v}) la position et la vitesse de la particule par rapport au centre du corps. Après déviation, la vitesse finale de la particule à l'infini est notée \vec{v}_f .



1. Montrez que le moment cinétique \vec{L} de la particule est constant.
2. Montrez que la particule frappera la planète, éventuellement en incidence rasante, si son paramètre d'impact vérifie $b \leq b_0$ avec

$$b_0 = R \left(1 + \frac{r_\infty}{R}\right)^{1/2}$$

Exprimez r_∞ en fonction de M et v_∞ .

Indice : Appliquez la conservation du moment cinétique et de l'énergie totale à une trajectoire rasante.

3. Montrez que le vecteur de Runge–Lenz $\vec{A} = \vec{v} \times \vec{L} - GMm \vec{r}/r$ est constant.

Indice : Nous rappelons que $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c}) \vec{b} - (\vec{a} \cdot \vec{b}) \vec{c}$.

4. Utilisez les résultats précédents pour montrer que, si $b > b_0$, l'angle de déviation D entre les vitesses initiales et finales de la particule est donné par

$$\tan\left(\frac{D}{2}\right) = \frac{r_\infty}{2b}$$

Considérons maintenant une lentille ayant un indice de réfraction égal à n . La face d'entrée plane est perpendiculaire à l'axe de symétrie Oz de la lentille. Des rayons lumineux parallèles à Oz frappent la face d'entrée. La lentille a une épaisseur variable $T(b)$, où b est la distance à l'axe optique, avec $T(b_0) = T_0$.

5. Trouvez comment $T(b)$ doit varier en fonction de b de sorte que, dans la limite des petites déviations, la déviation optique imposée par la lentille soit identique à la déviation gravitationnelle.

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